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Dynamics of Probability Distributions Over Classical Fields†

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Abstract

It is shown that the original Hopf formulation for the time evolution of probability distributions over classical fields follows deductively from the space-time version of the theory.

1. *Introduction*

Considerable interest is attached to the modern functional differential dynamical theory which describes the time evolution of a probability distribution over a statistical ensemble of classical fields. A generic functional differential equation for the characteristic functional Φ associated with a classical field-theoretic probability distribution was originally derived by Hopf (1952) and used by others (see Rosen, 1960), but an alternative space-time version of the probabilistic dynamical theory has been employed recently (Rosen, 1967, 1970). The purpose of the present article is to show how the original and alternative space-time formulations are related. We fix notation and clarify mathematical and physical ideas by first reviewing the original Hopf formulation and the more recently employed space-time version of the theory. Then we exhibit the formal relationship between the two alternative dscriptions for the dynamics of probability distributions over classical fields. It is shown that the original Hopf formulation follows deductively from the space-time version of the theory.

2. *Original Hopf Formulation*‡

The dynamical equations of the field theory are expressed in first-order form

$$\frac{\partial \phi}{\partial t} = Q[\phi] \tag{2.1}$$

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‡ See Hopf (1952) and Rosen (1960).

with the N real-valued field variables denoted by the N-tuple

$$\phi = (\phi_1(x), \ldots, \phi_N(x)) \tag{2.2}$$

and with the space-time coordinates abbreviated as

$$x = (\mathbf{x}, t) = (x_1, x_2, x_3, t) \tag{2.3}$$

Each of the N components of $Q[\phi] = (Q_1[\phi], \ldots, Q_N[\phi])$ in (2.1) is a functional of ϕ at the instant of time t, involving an arbitrary spatial transform and/or arbitrary spatial derivatives of ϕ. Although the form of $Q[\phi]$ may change with time through an explicit dependence on t, time derivatives of the components of ϕ are not supposed to appear in $Q[\phi]$. The definition of additional field variables may be required to bring the most general field theory which is local in the sense of time into the first order real-variable form (2.1).

To illuminate the generic notation in equations (2.1) and (2.2) by way of example, we cite the Navier–Stokes field theory for boundary-free incompressible fluid flow. Here we have $\phi \equiv \mathbf{u} = (u_1(x), u_2(x), u_3(x))$, the solenoidal velocity field, and the dynamical equations are cast in the form (2.1) as

$$\frac{\partial \mathbf{u}}{\partial t} = \nu \nabla^2 \mathbf{u} - (\mathbf{u} . \nabla \mathbf{u})^{\text{tr}} \equiv Q[\mathbf{u}] \tag{2.4}$$

with tr denoting the transverse (solenoidal) part of the inertial term. Compatible with equation (2.4) for all time, the incompressibility condition $\nabla . \mathbf{u} = 0$ need only be retained as an initial value requirement on the velocity field.

Let us now consider a statistical ensemble of field-theoretic dynamical systems governed by equation (2.1). If $dP_t[\phi(\mathbf{x})]$ denotes the probability measure assigned to $\phi(\mathbf{x})$ at t, we have

$$dP_t[\phi(\mathbf{x}, t)] = dP_0[\phi(\mathbf{x}, 0)] \tag{2.5}$$

where $\phi = \phi(\mathbf{x}, t)$ is the solution to (2.1) for $t \geqslant 0$ subject to the initial value $\phi(\mathbf{x}, 0)$. Thus the Gaussian probability measure

$$dP_t[\phi(\mathbf{x})] = \begin{pmatrix}\text{function}\\ \text{of } t\end{pmatrix} \Big[\exp\Big(-\tfrac{1}{2}\int\int \mathscr{D}_{\alpha\beta}(\mathbf{x}', \mathbf{x}'', t) \times \times \phi_\alpha(\mathbf{x}')\, \phi_\beta(\mathbf{x}'')\, d^3x'\, d^3x''\Big)\Big] \Big(\prod_{\gamma, \mathbf{x}} d\phi_\gamma(\mathbf{x})\Big) \tag{2.6}$$

with $\mathscr{D}_{\alpha\beta}(\mathbf{x}', \mathbf{x}'', t) = \mathscr{D}_{\beta\alpha}(\mathbf{x}'', \mathbf{x}', t)$ a real positive-definite symmetrical distribution (generalized function) is admissible for all $t \geqslant 0$ if and only if the dynamical equation (2.1) is linear,

$$\frac{\partial \phi_\alpha(\mathbf{x}, t)}{\partial t} = \int \mathscr{L}_{\alpha\beta}(\mathbf{x}, \mathbf{x}', t)\, \phi_\beta(\mathbf{x}', t)\, d^3x' \equiv Q_\alpha[\phi] \tag{2.7}$$

with $\mathscr{L}_{\alpha\beta}(\mathbf{x},\mathbf{x}',t)$ a generic real distribution.† In fact it follows from (2.5) that the symmetrical distribution in (2.6) is related to the distribution $\mathscr{L}_{\alpha\beta}(\mathbf{x},\mathbf{x}',t)$ in (2.7) by the dynamical equation

$$\frac{\partial \mathscr{D}_{\alpha\beta}(\mathbf{x}',\mathbf{x}'',t)}{\partial t} = -\int (\mathscr{L}_{\gamma\alpha}(\mathbf{x},\mathbf{x}',t)\,\mathscr{D}_{\gamma\beta}(\mathbf{x},\mathbf{x}'',t) + \mathscr{D}_{\alpha\gamma}(\mathbf{x}',\mathbf{x},t)\,\mathscr{L}_{\gamma\beta}(\mathbf{x},\mathbf{x}'',t))\,d^3x \tag{2.8}$$

On the other hand, if the dynamical equation (2.1) is non-linear [as exemplified by the Navier–Stokes equation (2.4)], a Gaussian probability measure (2.6) is inadmissible for all $t \geqslant 0$. We note that conservation of probability is expressed in the general case by

$$\int_{\text{all } \phi(\mathbf{x})} dP_t[\phi(\mathbf{x})] = 1 \tag{2.9}$$

a normalization condition which is manifestly compatible with (2.5). Ensemble averages of functionals of ϕ at t,

$$\begin{aligned}\langle F[\phi(\mathbf{x})]\rangle_t &\equiv \int_{\text{all } \phi(\mathbf{x})} F[\phi(\mathbf{x})]\,dP_t[\phi(\mathbf{x})] \\ &= \int_{\text{all } \phi(\mathbf{x},0)} F[\phi(\mathbf{x},t)]\,dP_0[\phi(\mathbf{x},0)]\end{aligned} \tag{2.10}$$

can be evaluated by functional integration‡ for a prescribed initial probability measure $dP_0[\phi(\mathbf{x})]$ if the general solution to equation (2.1) is available. In the event that the general solution to (2.1) is unavailable, we can proceed by introducing the Hopf characteristic functional associated with the probability distribution,

$$\begin{aligned}\Phi[y;t] &\equiv \left\langle \exp\left(i\int y_\alpha(\mathbf{x})\,\phi_\alpha(\mathbf{x})\,d^3x\right)\right\rangle_t \\ &= 1 + i\int y_\alpha(\mathbf{x})\,\langle\phi_\alpha(\mathbf{x})\rangle_t\,d^3x - \\ &\quad -\tfrac{1}{2}\int y_\alpha(\mathbf{x}')\,y_\beta(\mathbf{x}'')\,\langle\phi_\alpha(\mathbf{x}')\,\phi_\beta(\mathbf{x}'')\rangle_t\,d^3x'\,d^3x'' + \\ &\quad + \cdots\end{aligned} \tag{2.11}$$

a series which embraces all correlation functions with the real-valued functions

$$y = (y_1(\mathbf{x}),\ldots,y_N(\mathbf{x})) \tag{2.12}$$

† Local spatial derivative terms in $Q[\phi]$ appear in $\mathscr{L}_{\alpha\beta}(\mathbf{x},\mathbf{x}',t)$ as spatial derivatives on $\delta(\mathbf{x}-\mathbf{x}')$; for example, $Q[\phi] = (\text{const.})\,\nabla^2\phi$ yields $\mathscr{L}_{\alpha\beta}(\mathbf{x},\mathbf{x}',t) = (\text{const.})\,\delta_{\alpha\beta}\nabla^2\,\delta(\mathbf{x}-\mathbf{x}')$.

‡ For a recent introduction to field-theoretic functional integration methods see Rosen, G. (1969), *Formulations of Classical and Quantum Dynamical Theory*, Chap. 4. Academic Press, Inc., New York.

independent of t. For example, the probability measure (2.6) produces

$$\Phi[y;t] = \exp\left(-\tfrac{1}{2}\int\int \mathscr{F}_{\alpha\beta}(\mathbf{x}',\mathbf{x}'',t)\, y_\alpha(\mathbf{x}')\, y_\beta(\mathbf{x}'')\, d^3x'\, d^3x''\right) \qquad (2.13)$$

where the real positive-definitive symmetrical distribution $\mathscr{F}_{\alpha\beta}(\mathbf{x}',\mathbf{x}'',t)$ is the inverse of the distribution in (2.6),

$$\int \mathscr{F}_{\alpha\gamma}(\mathbf{x}',\mathbf{x},t)\, \mathscr{D}_{\gamma\beta}(\mathbf{x},\mathbf{x}'',t)\, d^3x = \delta_{\alpha\beta}\, \delta(\mathbf{x}'-\mathbf{x}'') \qquad (2.14)$$

In the general case, ensemble averages (2.10) are extracted from Φ by functional differentiation,

$$\langle F[\phi(\mathbf{x})]\rangle_t = \left(F\left[\frac{-i\delta}{\delta y(\mathbf{x})}\right]\Phi[y;t]\right)\bigg|_{y=0} \qquad (2.15)$$

Hence, ensemble averages at t are obtainable immediately from a closed-form expression for the characteristic functional at t. By virtue of (2.10) we have

$$\Phi[y;t] = \int_{\text{all } \phi(\mathbf{x},0)} \left[\exp\left(i\int y_\alpha(\mathbf{x})\, \phi_\alpha(\mathbf{x},t)\, d^3x\right)\right] dP_0[\phi(\mathbf{x},0)] \qquad (2.16)$$

and thus the dynamical evolution of the characteristic functional is given by the Hopf equation

$$\frac{\partial\Phi[y;t]}{\partial t} = i\int y_\alpha(\mathbf{x})\, Q_\alpha\left[\frac{-i\delta}{\delta y(\mathbf{x})}\right]\Phi[y;t]\, d^3x \qquad (2.17)$$

for a field governed by equation (2.1). It is easily verified that the Hopf equation (2.17) is generally compatible with properties of the characteristic functional manifest in the definition (2.11), such as

$$\Phi[0;t] \equiv 1 \qquad (2.18)$$

and

$$\Phi[y;t]^* \equiv \Phi[-y;t] \qquad (2.19)$$

In particular, for a linear theory with dynamical equations of the form (2.7) the Hopf equation (2.17) is satisfied by the characteristic functional (2.13) if the symmetrical distribution in (2.13) satisfies the dynamical equation implied by (2.8) and (2.14),

$$\frac{\partial\mathscr{F}_{\alpha\beta}(\mathbf{x}',\mathbf{x}'',t)}{\partial t} = \int (\mathscr{L}_{\alpha\gamma}(\mathbf{x}',\mathbf{x},t)\, \mathscr{F}_{\gamma\beta}(\mathbf{x},\mathbf{x}'',t) + \mathscr{F}_{\alpha\gamma}(\mathbf{x}',\mathbf{x},t)\, \mathscr{L}_{\beta\gamma}(\mathbf{x}'',\mathbf{x},t))\, d^3x \qquad (2.20)$$

3. *Space-Time Version of the Theory*†

We now consider an alternative way of assigning probability to the field-theoretic dynamical systems in the statistical ensemble. Namely, we

† See Rosen (1967, 1970).

assign a probability measure $d\mathbf{P}[\phi]$ to all space-time functions $\phi = \phi(x) = \phi(\mathbf{x}, t)$ with t in a prescribed (finite or semi-infinite) interval $0 \leqslant t \leqslant T$. The probability measure $d\mathbf{P}[\phi]$ is non-negative for a physically admissible field history $\phi = \phi(x)$ which satisfies equation (2.1) and $d\mathbf{P}[\phi]$ vanishes for all space-time functions $\phi = \phi(x)$ which do not satisfy (2.1) for $0 \leqslant t \leqslant T$. This property of $d\mathbf{P}[\phi]$ being concentrated on physically admissible $\phi = \phi(x)$ is expressed by

$$\left(\frac{\partial \phi(x)}{\partial t} - Q[\phi(x)]\right) d\mathbf{P}[\phi] = 0 \tag{3.1}$$

an equation holding for all space-time functions $\phi = \phi(x)$, all space points $\mathbf{x}$, and all values of t in the interval $0 \leqslant t \leqslant T$. A characteristic functional

$$\mathbf{\Phi}[\mu] \equiv \int_{\text{all}\,\phi(x)} \left[\exp\left(i \int_0^T \int \mu_\alpha(x)\, \phi_\alpha(x)\, dx\right)\right] d\mathbf{P}[\phi(x)] \tag{3.2}$$

where

$$\mu = (\mu_1(x), \ldots, \mu_N(x)) \tag{3.3}$$

are real-valued space-time functions and $dx \equiv d^3x\,dt$, is associated with the probability measure $d\mathbf{P}[\phi]$. It follows from (3.1) that the characteristic functional (3.2) is a solution to the equations

$$\left(i \frac{\partial}{\partial t} \frac{\delta}{\delta \mu(x)} + Q\left[-i \frac{\delta}{\delta \mu(x)}\right]\right) \mathbf{\Phi}[\mu] = 0 \tag{3.4}$$

for all component index values of the N-tuple functional differential operator in the parentheses and all admissible space-time points x. In the case of a linear theory with dynamical equations (2.7) the solution to (3.4) is

$$\mathbf{\Phi}[\mu] = \exp\left(-\tfrac{1}{2} \int_0^T \int \int_0^T \int \mathscr{K}_{\alpha\beta}(x', x'')\, \mu_\alpha(x')\, \mu_\beta(x'')\, dx'\, dx''\right) \tag{3.5}$$

in which the real positive-definite symmetrical distribution $\mathscr{K}_{\alpha\beta}(x', x'') \equiv \mathscr{K}_{\beta\alpha}(x'', x')$ satisfies the dynamical equation

$$\frac{\partial \mathscr{K}_{\alpha\beta}(x', x'')}{\partial t'} = \int \mathscr{L}_{\alpha\gamma}(\mathbf{x}', \mathbf{x}, t')\, \mathscr{K}_{\gamma\beta}(x, x'')\Big|_{t=t'} d^3x \tag{3.6}$$

Ensemble averages of functionals of $\phi = \phi(x)$ with t in the interval $0 \leqslant t \leqslant T$ are extracted from the characteristic functional $\mathbf{\Phi}[\mu]$ by the formula

$$\begin{aligned}\langle \mathbf{F}[\phi(x)] \rangle &\equiv \int_{\text{all}\,\phi(x)} \mathbf{F}[\phi(x)]\, d\mathbf{P}[\phi(x)] \\ &= \left(\mathbf{F}\left[\frac{-i\delta}{\delta \mu(x)}\right] \mathbf{\Phi}[\mu]\right)\Big|_{\mu=0}\end{aligned} \tag{3.7}$$

Obtainable immediately from a closed-form expression for the characteristic functional (3.2), the expectation values (3.7) for field quantities measured at times in the interval $0 \leqslant t \leqslant T$ are clearly more general than the expectation values (2.10) for field quantities measured at the instant of time t. Thus, for example, the generic space-time correlation functions $\langle \phi_\alpha(x')\, \phi_\beta(x'')\rangle$ with $0 \leqslant t', t'' \leqslant T$ are given by (3.7), while only the equal-time space correlation functions with $t' = t'' \equiv t$ are given by (2.10).

4. *Relationship Between the Alternative Formulations*

From the definitions in the preceding paragraphs it follows that

$$dP_t[\xi(\mathbf{x}')] = \int_{\text{all } \phi(x') \ni \phi(x')|_{t'=t} = \xi(\mathbf{x}')} d\mathbf{P}[\phi(x')] \tag{4.1}$$

Therefore the Hopf characteristic functional (2.11) is related to the characteristic functional (3.2) by

$$\Phi[y; t] = \mathbf{\Phi}[\hat{\mu}_t] \tag{4.2}$$

where

$$\hat{\mu}_t = \hat{\mu}_t(x') \equiv y(\mathbf{x}')\, \delta(t' - t) \tag{4.3}$$

To illustrate the general relationship provided by (4.2), consider the characteristic functional (3.5) for a linear theory; in this special case we have

$$\begin{aligned} \mathbf{\Phi}[\hat{\mu}_t] &= \exp\left(-\tfrac{1}{2}\int\int \mathscr{K}_{\alpha\beta}(x', x'')\Big|_{t'=t''=t} y_\alpha(\mathbf{x}')\, y_\beta(\mathbf{x}'')\, d^3x'\, d^3x''\right) \\ &= \Phi[y; t] \end{aligned} \tag{4.4}$$

and hence by using (2.13) we obtain

$$\mathscr{F}_{\alpha\beta}(\mathbf{x}', \mathbf{x}'', t) = \mathscr{K}_{\alpha\beta}(x', x'')\,|_{t'=t''=t} \tag{4.5}$$

an expression which is readily seen to satisfy equation (2.20) as a consequence of equation (3.6). Now the variation of (4.2) produces

$$\begin{aligned} \delta\Phi[y; t] &\equiv \int \frac{\delta\Phi[y; t]}{\delta y_\alpha(\mathbf{x}')}\, \delta y_\alpha(\mathbf{x}')\, d^3x' \\ &= \delta\mathbf{\Phi}[\hat{\mu}_t] \equiv \int \frac{\delta\mathbf{\Phi}[\mu]}{\delta\mu_\alpha(x')}\Big|_{\mu=\hat{\mu}_t} \delta\hat{\mu}_{t\alpha}(x')\, dx' \\ &= \int \frac{\delta\mathbf{\Phi}[\mu]}{\delta\mu_\alpha(x')}\Big|_{\substack{\mu=\hat{\mu}_t \\ t'=t}} \delta y_\alpha(\mathbf{x}')\, d^3x', \end{aligned} \tag{4.6}$$

and hence we have

$$\frac{\delta\Phi[y; t]}{\delta y(\mathbf{x}')} = \frac{\delta\mathbf{\Phi}[\mu]}{\delta\mu(x')}\Big|_{\substack{\mu=\hat{\mu}_t \\ t'=t}}. \tag{4.7}$$

Mathematical induction then confirms that higher-order functional derivatives of $\Phi[y; t]$ and $\mathbf{\Phi}[\mu]$ are related by the anticipated equations

$$\frac{\delta^n \Phi[y;t]}{\delta y(\mathbf{x}^{(1)}) \cdots \delta y(\mathbf{x}^{(n)})} = \left. \frac{\delta^n \mathbf{\Phi}[\mu]}{\delta \mu(x^{(1)}) \cdots \delta \mu(x^{(n)})} \right|_{t^{(1)}=\cdots=t^{(n)}=t}^{\mu=\hat{\mu}_t} \tag{4.8}$$

Differentiating (4.2) with respect to t, we find

$$\begin{aligned}
\frac{\partial \Phi[y;t]}{\partial t} &= \int_0^T \int \left. \frac{\delta \mathbf{\Phi}[\mu]}{\delta \mu_\alpha(x')} \right|_{\mu=\hat{\mu}_t} \frac{\partial}{\partial t}(y_\alpha(\mathbf{x}')\, \delta(t'-t))\, dx' \\
&= \int y_\alpha(\mathbf{x}') \left(\frac{\partial}{\partial t'} \frac{\delta \mathbf{\Phi}[\mu]}{\delta \mu_\alpha(x')} \right) \Bigg|_{\substack{\mu=\hat{\mu}_t \\ t'=t}} d^3x' \\
&= i \int y_\alpha(\mathbf{x}') \left(Q_\alpha \left[-i \frac{\delta}{\delta \mu(x')} \right] \mathbf{\Phi}[\mu] \right) \Bigg|_{\substack{\mu=\hat{\mu}_t \\ t'=t}} d^3x' \\
&= i \int y_\alpha(\mathbf{x}')\, Q_\alpha \left[-i \frac{\delta}{\delta y(\mathbf{x}')} \right] \Phi[y;t]\, d^3x'
\end{aligned} \tag{4.9}$$

In (4.9) we have used equations (3.4) and (4.8); by virtue of (4.9) we see that the Hopf equation (2.17) is an implication of (4.2) and the functional differential dynamical equations (3.4). Specializing (3.7) for a functional $\mathbf{F}[\phi(x)] = F[\phi(\mathbf{x}, t)]$ that depends exclusively on the field at the instant of time t, we find

$$\begin{aligned}
\langle F[\phi(\mathbf{x},t)] \rangle &= \int_{\text{all } \phi(x)} F[\phi(\mathbf{x},t)]\, d\mathbf{P}[\phi(x)] \\
&= \int_{\text{all } \phi(x)} F[\phi(\mathbf{x})]\, dP_t[\phi(\mathbf{x})] \equiv \langle F[\phi(\mathbf{x})] \rangle_t
\end{aligned} \tag{4.10}$$

where (2.10) is recalled. Hence, the expectation value formula (2.10) is an implication of (4.2) and the expectation value formula (3.7). We have thus shown that the original Hopf formulation follows deductively from the space-time version of the theory.

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